SPECIALIZATION OF MONODROMY GROUP AND ℓ-INDEPENDENCE

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ABSTRACT. Let E be an abelian scheme over a geometrically connected variety X defined over k, a finitely generated field over $\mathbb{Q}$. Let η be the generic point of X and $x \in X$ a closed point. If $\mathfrak{g}_l$ and $(\mathfrak{g}_l)_x$ are the Lie algebras of the l-adic Galois representations for abelian varieties E_η and E_x, then $(\mathfrak{g}_l)_x$ is embedded in $\mathfrak{g}_l$ by specialization. We prove that the set $\{x \in X \text{ closed point} \mid (\mathfrak{g}_l)_x \subsetneq \mathfrak{g}_l\}$ is independent of l and confirm Conjecture 5.5 in [2].

§0. Introduction

Let E be an abelian scheme of relative dimension n over a geometrically connected variety X defined over k, a finitely generated field over $\mathbb{Q}$. If K is the function field of X and η is the generic point of X, then $A := E_\eta$ is an abelian variety of dimension n defined over K. The structure morphism $X \to \mathrm{Spec}(k)$ induces at the level of *étale* fundamental groups a short exact sequence of profinite groups:

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \Gamma_k := \mathrm{Gal}(\overline{k}/k) \to 1. \tag{0.1}$$

Any closed point $x : \mathrm{Spec}(\mathbf{k}(x)) \to X$ induces a splitting $x : \Gamma_{\mathbf{k}(x)} \to \pi_1(X_{\mathbf{k}(x)})$ of equation (0.1) for $\pi_1(X_{\mathbf{k}(x)})$.

Let $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. For each prime number l, we have the Galois representation $\rho_l : \Gamma_K \to \mathrm{GL}(T_l(A))$ where $T_l(A)$ is the l-adic Tate module of A. This representation is unramified over X and factors through $\rho_l : \pi_1(X) \to \mathrm{GL}(T_l(A))$ (still denote the map by ρ_l for simplicity). The image of ρ_l is a compact l-adic Lie subgroup of $\mathrm{GL}(T_l(A)) \cong \mathrm{GL}_{2n}(\mathbb{Z}_l)$. Any closed point $x : \mathrm{Spec}(\mathbf{k}(x)) \to X$ induces an l-adic Galois representation by restricting ρ_l to $x(\Gamma_{\mathbf{k}(x)})$. This representation is isomorphic to the Galois representation of $\Gamma_{\mathbf{k}(x)}$ on the l-adic Tate module of E_x, the abelian variety specialized at x.

For simplicity, write $G_l := \rho_l(\pi_1(X))$, $\mathfrak{g}_l := \mathrm{Lie}(G_l)$, $(G_l)_x := \rho_l(x(\Gamma_{\mathbf{k}(x)}))$ and $(\mathfrak{g}_l)_x := \mathrm{Lie}((G_l)_x)$. We have $(\mathfrak{g}_l)_x \subset \mathfrak{g}_l$. We set X^{cl} the set of closed points of X and define the exceptional set

$$X_{\rho_{E,l}} := \{x \in X^{cl} | (\mathfrak{g}_l)_x \subsetneq \mathfrak{g}_l\}.$$

The main result (Theorem 1.4) of this note is that the exceptional set $X_{\rho_{E,l}}$ is independent of l. Conjecture 5.5 in [Cadoret & Tamagawa 2] is then a direct application of our theorem.

§1. l-independence of $X_{\rho_{E,l}}$

Theorem 1.1. (Serre [5 §1]) Let A be an abelian variety defined over a field K finitely generated over $\mathbb{Q}$ and let $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$. If $\rho_l : \Gamma_K \to \mathrm{GL}(T_l(A))$ is the l-adic representation of Γ_K, then the Lie algebra $\mathfrak{g}_l$ of $\rho_l(\Gamma_K)$ is algebraic and the rank of $\mathfrak{g}_l$ is independent of the prime l.

Since $V_l := T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is a semisimple Γ_K-module (Faltings and Wüstholz [3 Chap. 6]), the action of the Zariski closure of $\rho_l(\Gamma_K)$ in GL_{V_l} is also semisimple on V_l. Therefore it is a reductive algebraic group (Borel [1]). By Theorem 1.1, $\mathfrak{g}_l$ is algebraic. So the rank of $\mathfrak{g}_l$ is just the dimension of maximal tori. We state two more theorems:

Theorem 1.2. (Faltings and Wüstholz [3 Chap. 6]) Let A be an abelian variety defined over a field k finitely generated over $\mathbb{Q}$ and let $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$. Then the map $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to \mathrm{End}_{G_k}(V_l(A))$ is an isomorphism.

Theorem 1.3. (Zarhin [6 §5]) Let V be a finite dimensional vector space over a field of characteristic 0. Let $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathrm{End}(V)$ be Lie algebras of reductive subgroups of GL_V. We assume that the centralizers of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ in $\mathrm{End}(V)$ are equal and that the ranks of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are equal. Then $\mathfrak{g}_1 = \mathfrak{g}_2$.

We are now able to prove our main theorem.

Theorem 1.4. The set $X_{\rho_{E,l}}$ is independent of l.

Proof. Suppose $x \in X^{cl} \backslash X_{\rho_l}$, then $(\mathfrak{g}_l)_x = \mathfrak{g}_l$. It suffices to show $\mathfrak{g}_{l'} = (\mathfrak{g}_{l'})_x := \mathrm{Lie}(\rho_{l'}(x(\Gamma_{\mathbf{k}(x)})))$ for any prime number l'. Since base change with finite field extension of $\mathbf{k}(x)$ does not change the Lie algebras, $\mathrm{End}_{\overline{k}}(E_x)$ is finitely generated, and we have the exponential map from Lie algebras to Lie groups, we may assume that $\mathrm{End}_{\overline{k}}(E_x) = \mathrm{End}_k(E_x)$ and $\mathrm{End}_{\Gamma_k}(V_l(E_x)) = \mathrm{End}_{(\mathfrak{g}_l)_x}(V_l(E_x))$. We do the same for the abelian variety E_η/K. We therefore have

$$\dim_{\mathbb{Q}_{l'}}(\mathrm{End}_{\mathfrak{g}_{l'}}(V_p(E_\eta))) \overset{1}{=} \dim_{\mathbb{Q}_{l'}}(\mathrm{End}_K(E_\eta) \otimes_{\mathbb{Z}} \mathbb{Q}_{l'})$$

$$\overset{2}{=} \dim_{\mathbb{Q}_l}(\mathrm{End}_K(E_\eta) \otimes_{\mathbb{Z}} \mathbb{Q}_l) \overset{3}{=} \dim_{\mathbb{Q}_l}(\mathrm{End}_{\mathfrak{g}_l}(V_l(E_\eta)))$$

$$\overset{4}{=} \dim_{\mathbb{Q}_l}(\mathrm{End}_{(\mathfrak{g}_l)_x}(V_l(E_x))) \overset{5}{=} \dim_{\mathbb{Q}_l}(\mathrm{End}_k(E_x) \otimes_{\mathbb{Z}} \mathbb{Q}_l)$$

$$\overset{6}{=} \dim_{\mathbb{Q}_{l'}}(\mathrm{End}_k(E_x) \otimes_{\mathbb{Z}} \mathbb{Q}_p) \overset{7}{=} \dim_{\mathbb{Q}_{l'}}(\mathrm{End}_{(\mathfrak{g}_{l'})_x}(V_{l'}(E_x))).$$

Theorem 1.2 implies the first, third, fifth and seventh equality. The dimensions of $\mathrm{End}_K(E_\eta) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ and $\mathrm{End}_k(E_x) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ as vector spaces are independent of l imply the second and the sixth equality. $\mathfrak{g}_l = (\mathfrak{g}_l)_x$ implies the fourth equality.

We have $\mathrm{End}_{\mathfrak{g}_{l'}}(V_{l'}(E_\eta)) = \mathrm{End}_{(\mathfrak{g}_{l'})_x}(V_{l'}(E_x))$ because the left one is contained in the right one. In other words, the centralizer of $(\mathfrak{g}_{l'})_x$ is equal to the centralizer of $\mathfrak{g}_{l'}$. We know that $(\mathfrak{g}_{l'})_x \subset \mathfrak{g}_{l'}$ are both reductive from the semisimplicity of Galois representaion (Faltings and

Wüstholz [3 Chap. 6]). By Theorem 1.1 on l-independence of reductive ranks and $\mathfrak{g}_l = (\mathfrak{g}_l)_x$, we have:

$$\mathrm{rank}(\mathfrak{g}_{l'}) = \mathrm{rank}(\mathfrak{g}_l) = \mathrm{rank}(\mathfrak{g}_l)_x = \mathrm{rank}(\mathfrak{g}_{l'})_x.$$

Therefore, by Theorem 1.3 we conclude that $(\mathfrak{g}_{l'})_x = \mathfrak{g}_{l'}$ and thus prove the theorem. $\square$

Corollary 1.5 (Conjecture 5.5 [2]). Let k be a field finitely generated over $\mathbb{Q}$, X a smooth, separated, geometrically connected curve over k with quotient field K. Let η be the generic point of X and E an abelian scheme over X. Let $\rho_l : \pi_1(X) \to \mathrm{GL}(T_l(E_\eta))$ be the l-adic representation. Then there exists a finite subset $X_E \subset X(k)$ such that for any prime l, $X_{\rho_{E,l}} = X_E$, where $X_{\rho_{E,l}}$ is the set of all $x \in X(k)$ such that $(G_l)_x$ is not open in $G_l := \rho_l(\pi_1(X))$.

Proof. The uniform open image theorem for GSRP representations [2 Thm. 1.1] implies the finiteness of $X_{\rho_{E,l}}$. Theorem 1.4 implies l-independence. $\square$

Corollary 1.6. Let A be an abelian variety of dimension $n \geq 1$ defined over a field K finitely generated over $\mathbb{Q}$. Let $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ denote the absolute Galois group of K. For each prime number l, we have the Galois representation $\rho_l : \Gamma_K \to \mathrm{GL}(T_l(A))$ where $T_l(A)$ is the l-adic Tate module of A. If the Mumford-Tate conjecture for abelian varieties over number fields is true, then there is an algebraic subgroup H of $\mathbf{GL}_{2n}$ defined over $\mathbb{Q}$ such that $\rho_l(\Gamma_K)^\circ$ is open in $H(\mathbb{Q}_l)$ for all l.

Proof. There exists an abelian scheme E over a variety X defned over a number field k such that the function field of X is K and $E_\eta = A$ where η is the generic point of X (see, e.g., Milne [4 §20]). By [5 §1], there exists a closed point $x \in X$ such that $(\mathfrak{g}_l)_x = \mathfrak{g}_l$. Therefore, we have $(\mathfrak{g}_l)_x = \mathfrak{g}_l$ for any prime l by Theorem 1.4. Since all Lie algebras are algebraic (Theorem 1.1), if we take H as the Mumford-Tate group of E_x, $\rho_l(\Gamma_K)^\circ$ is then open in $H(\mathbb{Q}_l)$ for all l. $\square$

Question. Is the algebraic group H in Corollary 1.6 isomorphic to the Mumford-Tate group of the abelian variety A?

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